

ON THE NUMBER OF STABLE QUIVER REPRESENTATIONS OVER FINITE FIELDS

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ABSTRACT. We prove a new formula for the generating function of polynomials counting absolutely stable representations of quivers over finite fields. The case of irreducible representations is studied in more detail.

1. INTRODUCTION

Many objects arising in representation-theoretic contexts have the favourable property that they are counted over finite fields by evaluating polynomials with integer coefficients at the cardinality of the field. In particular, this applies to the rational points of many moduli spaces of representations, as well as other varieties appearing in representation theory (see [6, Appendix] or [15] for a general discussion of varieties whose rational points over finite fields are counted by polynomials).

This special property opens the very interesting possibility to prove, or at least predict, topological and geometric properties of such varieties from the polynomials arising in this way. For example, the Euler characteristic of such varieties can be computed as the evaluation of the counting polynomial at 1. It is also typical that constant term conjectures or positivity conjectures on the polynomials can be formulated, often with very interesting conjectural geometric interpretations. A famous example is provided by the Kac conjecture, which interprets the constant terms of the polynomials counting absolutely indecomposable quiver representations of fixed dimension vector as root multiplicities in corresponding Kac-Moody algebras (this conjecture was proved in [2] in the case of indivisible dimension vectors; a proof in the general case was announced in [5]). Furthermore, the mere existence of these polynomials might substantiate conjectures on rationality of varieties or the existence of cell decompositions.

The second named author proved the polynomiality property for the number of isomorphism classes of absolutely stable (in particular, irreducible) quiver representations, equivalently, rational points over finite fields of moduli of stable quiver representations in [12], using Hall algebra techniques and the arithmetic of representations. However, the resulting formula for these polynomials is of a recursive nature, and it is highly complicated. No special properties of the polynomials could be extracted, except for some predictions based on computer experiments.

The first named author already established the usefulness of λ -ring techniques for the study of counting polynomials in [8], thereby deriving a combinatorial reformulation of the Kac conjecture mentioned above.

In the present paper both approaches are combined, by implementing λ -ring techniques into the methods of [12]. Our first result, Theorem 4.1, provides a simple formula for the generating function of the polynomials counting absolutely stable

quiver representations, in terms of certain rational functions introduced in [11] (and also used in [3]). Our second result, Theorem 5.1, gives a further simplification of the formula when the irreducible representations are counted.

In the latter case, computer experiments suggest to consider the expansion of the counting polynomials at $q = 1$. In [12], an explicit formula was conjectured for the linear term of this expansion (the constant term being zero in general). This conjecture was proved in [10] using entirely different methods, namely, fixed point localization in quiver moduli. Our third result, Theorem 6.1 and its corollary, give a new formula for the linear term, and rederive the result of [10] in the case of multiple loop quivers.

Finally, we conjecture that the Taylor coefficients around $q = 1$ are nonnegative. Based on further computer experiments, a conjecture on these Taylor coefficients is formulated in Remark 6.5. We hope that a proof of this conjecture, and a combinatorial interpretation of the Taylor coefficients, could give detailed information on the geometry of moduli spaces of irreducible quiver representations.

The paper is organized as follows: Section 2 discusses the relevant λ -ring methods, and Section 3 collects the necessary facts on (stable) quiver representations and Hall algebras. The first main result, the formula for the generating function of polynomials counting absolutely stable representations, is derived in Section 4, and specialized to the case of irreducible representations in Section 5. The final Section 6 gives a recursive formula for the latter case and rederives the formula for the linear term of the counting polynomials in the case of multiple loop quiver.

2. COMBINATORIAL PREREQUISITES

We endow the ring of power series $R = \mathbb{Q}(q)[[x_1, \dots, x_r]]$ with a structure of a λ -ring (see [8, Appendix]) in terms of Adams operations by

$$\psi_n(f(q, x_1, \dots, x_r)) = f(q^n, x_1^n, \dots, x_r^n).$$

As a λ -ring, the ring R possesses also λ -operations and σ -operations. Let \mathfrak{m} be the unique maximal ideal of the ring R . We define the map $\text{Exp} : \mathfrak{m} \rightarrow 1 + \mathfrak{m}$ by the formula

$$\text{Exp}(f) = \sum_{k \geq 0} \sigma_k(f) = \exp \left(\sum_{k \geq 1} \frac{1}{k} \psi_k(f) \right).$$

This map has an inverse $\text{Log} : 1 + \mathfrak{m} \rightarrow \mathfrak{m}$, given by the Cadogan formula [1, 4, 8]

$$\text{Log}(f) := \sum_{k \geq 1} \frac{\mu(k)}{k} \psi_k(\log(f)),$$

where μ is a classical Möbius function. We define the map $\text{Pow} : (1 + \mathfrak{m}) \times R \rightarrow 1 + \mathfrak{m}$ by the formula

$$\text{Pow}(f, g) := \text{Exp}(g \text{Log}(f)).$$

The usual power map is defined by $f^g = \exp(g \log(f))$ for $f \in 1 + \mathfrak{m}$ and $g \in R$.

The following useful technical result relates the map Pow with the usual power map.

Lemma 2.1 ([8, Lemma 22]). *Let $f \in 1 + \mathfrak{m}$ and $g \in R$. Define the elements $g_d \in R$, $d \geq 1$ by the formula $\sum_{d|n} d \cdot g_d = \psi_n(g)$, $n \geq 1$. Then we have*

$$\text{Pow}(f, g) = \prod_{d \geq 1} \psi_d(f)^{g_d}.$$

Remark 2.2. The maps Exp, Log and Pow can be defined for an arbitrary complete λ -ring, see [8, Appendix]. Their basic properties can be also found there.

For $n \in \mathbb{Z}$ and $m \geq 0$, define the q-binomial coefficients (see [9])

$$[n, m] = \frac{\prod_{i=1}^m (1 - q^{n+i})}{\prod_{i=1}^m (1 - q^i)} \in \mathbb{Q}(q),$$

$$[\infty, m] = \frac{1}{\prod_{i=1}^m (1 - q^i)} \in \mathbb{Q}(q).$$

Let $I = \{1, \dots, r\}$ be a finite set. For any $\lambda \in \mathbb{Z}^I$ and $\alpha \in \mathbb{N}^I$, define

$$[\lambda, \alpha] := \prod_{i \in I} [\lambda^i, \alpha^i], \quad [\infty, \alpha] := \prod_{i \in I} [\infty, \alpha^i].$$

Define

$$p = p(x_1, \dots, x_r) = \sum_{\alpha \in \mathbb{N}^I} [\infty, \alpha] x^\alpha \in \mathbb{Q}(q)[[x_1, \dots, x_r]].$$

For any $\lambda \in \mathbb{Z}^I$, define

$$p(\lambda) = p(\lambda; x_1, \dots, x_r) = \sum_{\alpha \in \mathbb{N}^I} [\lambda, \alpha] x^\alpha \in \mathbb{Q}(q)[[x_1, \dots, x_r]].$$

As a particular case of the Heine formula [7, Theorem 13.1], one can show for $r = 1$

Lemma 2.3 (see [9]). *We have*

$$p(x) = \text{Exp} \left(\frac{x}{1-q} \right)$$

and, for any $n \in \mathbb{Z}$,

$$p(n; x) = \text{Exp} \left(\frac{1 - q^{n+1}}{1-q} x \right).$$

This lemma implies that

$$p(x_1, \dots, x_r) = \prod_{i \in I} \left(\sum_{k \geq 0} [\infty, k] x_i^k \right) = \prod_{i \in I} p(x_i) = \text{Exp} \left(\frac{\sum_{i \in I} x_i}{1-q} \right).$$

The functions p and $p(\lambda)$ can be related as follows. For any $f \in R$, define the conjugation $\overline{f(q)} := f(q^{-1})$. For any $\lambda \in \mathbb{Z}^I$, define the algebra homomorphism $S_\lambda : R \rightarrow R$ by $S_\lambda(x^\alpha) = q^{\sum \lambda^i \alpha^i} x^\alpha$.

Lemma 2.4 (see [9]). *For any $\lambda \in \mathbb{Z}^I$, we have $p(\lambda) = p \cdot S_\lambda(\overline{p})$.*

3. QUIVER REPRESENTATIONS

Let (Γ, I) be a finite quiver. Here Γ denotes the set of arrows and I denotes the set of vertices of the quiver. Usually we denote the quiver just by Γ . For any field k , we denote by $k\Gamma$ the path algebra of Γ over k . Any representation M of $k\Gamma$ can be decomposed into a sum of k -vector spaces $M = \bigoplus_{i \in I} M_i$ in a natural way. We define the dimension of M to be $\dim M = (\dim M_i)_{i \in I} \in \mathbb{N}^I$.

We define the Ringel form on \mathbb{Z}^I to be the bilinear form defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in I} \alpha^i \beta^i - \sum_{h: i \rightarrow j} \alpha^i \beta^j,$$

for any $\alpha = (\alpha^i)_{i \in I} \in \mathbb{Z}^I$ and $\beta = (\beta^i)_{i \in I} \in \mathbb{Z}^I$. Define the Tits form on \mathbb{Z}^I to be the quadratic form on \mathbb{Z}^I defined by $T(\alpha) = \langle \alpha, \alpha \rangle$, for any $\alpha = (\alpha^i)_{i \in I} \in \mathbb{Z}^I$.

Define the space of $k\Gamma$ -representations of dimension $\alpha \in \mathbb{N}^I$ to be

$$\mathcal{R}_\alpha(k) = \bigoplus_{h: i \rightarrow j} \text{Hom}_k(k^{\alpha^i}, k^{\alpha^j}).$$

The group $\text{GL}_\alpha(k) = \prod_{i \in I} \text{GL}_{\alpha^i}(k)$ acts on $\mathcal{R}_\alpha(k)$ by

$$(g_i)_{i \in I} \cdot (x_h)_{h \in \Gamma} = (g_j x_h g_i^{-1})_{h: i \rightarrow j}$$

and the orbits of this action parametrize the isomorphism classes of $k\Gamma$ -representations of dimension α .

3.1. Stable representations. We define a stability map (or just stability) to be a map $\theta : I \rightarrow \mathbb{Z}$. It can be extended by linearity to $\theta : \mathbb{Z}^I \rightarrow \mathbb{Z}$. Associated with θ , we define a slope function $\mu : \mathbb{N}^I \setminus \{0\} \rightarrow \mathbb{Q}$ by

$$\mu(\alpha) = \frac{\theta(\alpha)}{\text{ht } \alpha},$$

where $\text{ht } \alpha = \sum_{i \in I} \alpha^i$. For any nonzero $k\Gamma$ -representation M , we define $\mu(M) = \mu(\dim M)$.

Let us fix once and for all some stability θ . A $k\Gamma$ -representation M is called stable (respectively, semistable) if for any nonzero proper subrepresentation $N \subset M$, we have $\mu(N) < \mu(M)$ (resp. $\mu(N) \leq \mu(M)$). A stable representation of $k\Gamma$ is called absolutely stable if it stays stable after any finite field extension of k . A semistable representation is called polystable if it is a direct sum of stable representations.

For any $\mu \in \mathbb{Q}$, we denote by $\text{mod}_\mu k\Gamma$ the category of semistable $k\Gamma$ -representations having slope μ (by convention, the zero representation is also contained in this category). The category $\text{mod}_\mu k\Gamma$ is an abelian category and its simple objects are precisely the stable representations. The subcategory of $\text{mod}_\mu k\Gamma$ consisting of stable representations is denoted by $\text{mod}_\mu^s k\Gamma$.

Let $\alpha \in \mathbb{N}^I$. Define $\mathcal{R}_\alpha^{ss}(k) \subset \mathcal{R}_\alpha(k)$ to be the subset of semistable representations. It is invariant under the action of $\text{GL}_\alpha(k)$ and the orbits of this action parametrize the isomorphism classes of semistable representations of dimension α . For any finite field \mathbb{F}_q , we denote by $r_\alpha(q)$ the number

$$(1) \quad r_\alpha(q) = \frac{\#\mathcal{R}_\alpha^{ss}(\mathbb{F}_q)}{\#\text{GL}_\alpha(\mathbb{F}_q)}.$$

It was proved in [11] that the $r_\alpha(q)$ are rational functions in the variable q . More precisely, we have

Proposition 3.1 (see [11, Corollary 6.2]). *For any $\alpha \in \mathbb{N}^I \setminus \{0\}$, we have*

$$r_\alpha(q) = \sum_{(\alpha_1, \dots, \alpha_k)} (-1)^{k-1} q^{-\sum_{i < j} \langle \alpha_i, \alpha_j \rangle} \prod_{i=1}^k \frac{\#\mathcal{R}_{\alpha_k}(\mathbb{F}_q)}{\#\mathrm{GL}_{\alpha_k}(\mathbb{F}_q)},$$

where the sum runs over all tuples $(\alpha_1, \dots, \alpha_k)$ of vectors in $\mathbb{N}^I \setminus \{0\}$, such that $\sum_{i=1}^k \alpha_i = \alpha$ and $\mu(\sum_{i=1}^j \alpha_i) > \mu(\alpha)$ for any $1 \leq j < k$.

The goal of the paper is to provide a simple formula expressing the numbers of absolutely stable representations in terms of the rational functions r_α .

3.2. Absolutely stable representations. Given a $k\Gamma$ -representation S , we define $r_S = \dim_k \mathrm{End}(S)$. A stable representation S is absolutely stable if and only if $r_S = 1$, see [12, Proposition 4.4].

Let $\alpha \in \mathbb{N}^I$ and $r \geq 1$. Given a finite field \mathbb{F}_q , we denote by $a_\alpha(q)$ the number of isomorphism classes of absolutely stable $\mathbb{F}_q\Gamma$ -representations of dimension α . We denote by $s_{\alpha,r}(q)$ the number of isomorphism classes of stable $\mathbb{F}_q\Gamma$ -representations S of dimension α and with $r_S = r$. It was proved in [12] that $a_\alpha(q)$ and $s_{\alpha,r}(q)$ are polynomials in q , the first one having integer coefficients. These polynomials are related with each other as follows

Lemma 3.2 (cf. [12, Proposition 4.5]). *For any $\alpha \in \mathbb{N}^I$ and any $r \geq 1$, we have*

$$\psi_r(a_\alpha) = \sum_{k|r} ks_{k\alpha,k}.$$

Proof. According to [12, Proposition 4.5], we have

$$s_{\alpha,r}(q) = \frac{1}{r} \sum_{k|r} \mu(r/k) a_{\alpha/r}(q^k),$$

Using the Adams operations, we can rewrite this formula in the form

$$rs_{r\alpha,r} = \sum_{k|r} \mu(r/k) \psi_k(a_\alpha).$$

The Möbius inversion formula implies then the statement of the lemma. \square

Applying the above lemma together with Lemma 2.1 we get

Corollary 3.3. *Let \mathfrak{m} be the maximal ideal of $\mathbb{Q}(q)[[x_i | i \in I]]$ and let $f \in 1 + \mathfrak{m}$. Then*

$$\mathrm{Pow}(f, a_\alpha) = \prod_{r \geq 1} \psi_r(f)^{s_{r\alpha,r}}.$$

3.3. Hall algebra. Let k be a finite field. One defines the Hall algebra (see [14]) $H(k\Gamma)$ to be a \mathbb{Q} -vector space with a basis consisting of isomorphism classes of $k\Gamma$ -representations. For any two representations M, N of $k\Gamma$ the multiplication $[M] \cdot [N]$ is defined by

$$[M] \cdot [N] = \sum_{[X]} g_{M,N}^X [X],$$

where $g_{M,N}^X$ denotes the number of subrepresentations $U \subset X$ such that $U \simeq N$ and $X/U \simeq M$.

The Hall algebra $H(k\Gamma)$ possesses a grading given by the height of the dimension of representations. The corresponding completion of $H(k\Gamma)$ is denoted by $H((k\Gamma))$.

It is a local algebra and its invertible elements are those having nonzero component of degree zero.

Endow the algebra $\mathbb{Q}(q)[x_i | i \in I]$ with a new product defined by

$$x^\alpha \circ x^\beta = q^{-\langle \alpha, \beta \rangle} x^{\alpha+\beta}.$$

The new algebra is denoted by $\mathbb{Q}(q)^{\text{tw}}[x_i | i \in I]$. Given a finite field \mathbb{F}_q , we endow the algebra $\mathbb{Q}[x_i | i \in I]$ with a new product by the same formula as above, with q considered now as a rational number. The new algebra is denoted by $\mathbb{Q}_{\mathbb{F}_q}^{\text{tw}}[x_i | i \in I]$ or just by $\mathbb{Q}_q^{\text{tw}}[x_i | i \in I]$. The algebra $\mathbb{Q}(q)^{\text{tw}}[x_i | i \in I]$ (resp. $\mathbb{Q}_q^{\text{tw}}[x_i | i \in I]$) has a grading given by $\deg x^\alpha = \text{ht } \alpha$. The corresponding completion is denoted by $\mathbb{Q}(q)^{\text{tw}}[[x_i | i \in I]]$ (resp. $\mathbb{Q}_q^{\text{tw}}[[x_i | i \in I]]$). As a vector space it is isomorphic to $\mathbb{Q}(q)[[x_i | i \in I]]$ (resp. $\mathbb{Q}[[x_i | i \in I]]$).

Lemma 3.4 (see [12, Lemma 3.3]). *Let k be a finite field. Then the map $\int : H(k\Gamma) \rightarrow \mathbb{Q}_k^{\text{tw}}[x_i | i \in I]$ given by*

$$\int [M] = \frac{1}{\# \text{Aut } M} x^{\dim M}$$

is a homomorphism of graded algebras. It induces a homomorphism of completions $\int : H((k\Gamma)) \rightarrow \mathbb{Q}_k^{\text{tw}}[[x_i | i \in I]]$.

Let μ be a rational number and let k be a finite field. Define the element

$$e_\mu = e_{\mu, k} = \sum_{[M] \in \text{mod}_\mu k\Gamma} [M]$$

in the completed Hall algebra $H((k\Gamma))$. Its component of degree zero is nonzero and therefore e_μ is invertible.

Lemma 3.5 (see [12, Lemma 3.4]). *We have*

$$e_{\mu, k}^{-1} = \sum_{[M] \in \text{mod}_\mu k\Gamma} \gamma_M [M]$$

where

$$\gamma_M = \begin{cases} 0 & \text{if } M \text{ is not polystable,} \\ \prod_{[S] \in \text{mod}_\mu^s k\Gamma} (-1)^{m_S} (\# \text{End } S)^{\binom{m_S}{2}} & \text{if } M = \bigoplus_{[S] \in \text{mod}_\mu^s k\Gamma} S^{m_S}. \end{cases}$$

Lemma 3.6 (see [12]). *Let k be a finite field. Then*

$$\int e_{\mu, k} = \sum_{\alpha \in \mathbb{N}^I} r_\alpha(\#k) x^\alpha.$$

In the next section, we will express $\int e_\mu^{-1}$ in terms of the polynomials a_α , drastically simplifying the corresponding formula from [12]. This will allow us to relate the polynomials a_α with the rational functions r_α and to get a nice formula for the computation of a_α .

4. COMPUTATION OF THE INTEGRAL

Let (Γ, I) be a finite quiver, $\theta : I \rightarrow \mathbb{Z}$ be a stability and μ be a rational number. In this section, the slope function and the stability condition are always determined by θ . Representations are assumed to have slope μ , if not stated otherwise.

Let us define the generating functions

$$a(q) = \sum_{\alpha \in \mathbb{N}^I} a_\alpha(q) x^\alpha, \quad r(q) = \sum_{\alpha \in \mathbb{N}^I} r_\alpha(q) x^\alpha$$

in $\mathbb{Q}(q)[[x_i | i \in I]]$, where the polynomials $a_\alpha \in \mathbb{Z}[q]$ and the rational functions $r_\alpha \in \mathbb{Q}(q)$ were defined in Section 3.

Theorem 4.1. *We have*

$$r(q) \circ \text{Exp} \left(\frac{a(q)}{1-q} \right) = 1$$

in the algebra $\mathbb{Q}(q)^{\text{tw}}[[x_i | i \in I]]$.

Proof. It is enough to prove the corresponding formula for the specializations of q to the numbers of points of finite fields, i.e., for any finite field k , we have to prove

$$r(q)|_{q=\#k} \circ \text{Exp} \left(\frac{a(q)}{1-q} \right) \Big|_{q=\#k} = 1$$

in $\mathbb{Q}_k^{\text{tw}}[[x_i | i \in I]]$.

By Lemma 3.6, we have $\int e_{\mu,k} = r(q)|_{q=\#k}$. By Lemma 3.4, we have

$$\int e_{\mu,k} \circ \int e_{\mu,k}^{-1} = \int (e_{\mu,k} \cdot e_{\mu,k}^{-1}) = 1.$$

Thus, we just have to prove

$$\int e_{\mu,k}^{-1} = \text{Exp} \left(\frac{a(q)}{1-q} \right) \Big|_{q=\#k}.$$

This is the content of the next theorem. \square

Theorem 4.2. *Let k be a finite field. Then*

$$\int e_{\mu,k}^{-1} = \text{Exp} \left(\frac{a(q)}{1-q} \right) \Big|_{q=\#k}.$$

Proof. By Lemma 3.5, we have

$$\int e_{\mu,k}^{-1} = \sum_{[M]} \frac{\gamma_M}{\# \text{Aut } M} x^{\dim M} = \sum_{(m_S)_{[S]}} \frac{\prod_{[S]} (-1)^{m_S} (\# \text{End } S)^{\binom{m_S}{2}}}{\# \text{Aut} \left(\bigoplus_{[S]} S^{m_S} \right)} x^{\sum_{[S]} m_S \dim S}.$$

Note that

$$\text{Aut} \left(\bigoplus_{[S]} S^{m_S} \right) \simeq \prod_{[S]} \text{GL}_{m_S}(\text{End } S)$$

and therefore

$$\begin{aligned} \int e_{\mu,k}^{-1} &= \sum_{(m_S)_{[S]}} \prod_{[S]} \left(\frac{(-1)^{m_S} (\# \text{End } S)^{\binom{m_S}{2}}}{\# \text{GL}_{m_S}(\text{End } S)} x^{m_S \dim S} \right) \\ &= \prod_{[S]} \left(\sum_{m \geq 0} \frac{(-1)^m (\# \text{End } S)^{\binom{m}{2}}}{\# \text{GL}_m(\text{End } S)} x^{m \dim S} \right). \end{aligned}$$

For any stable representation S , the ring of endomorphisms $\text{End } S$ is a finite field and, for any finite field \mathbb{F}_s , we have

$$\frac{(-1)^m s^{\binom{m}{2}}}{\#\text{GL}_m(\mathbb{F}_s)} = \prod_{i=1}^m (1 - s^i)^{-1} = [\infty, m] \Big|_{q=s}.$$

It follows

$$\int e_{\mu,k}^{-1} = \prod_{[S]} \left(\sum_{m \geq 0} [\infty, m] x^{m \dim S} \Big|_{q=\#\text{End } S} \right).$$

Recall, that for any representation S , we have defined the integer r_S to be $\dim_k \text{End } S$. It follows that $\#\text{End } S = (\#k)^{r_S}$ and therefore

$$[\infty, m] \Big|_{q=\#\text{End } S} = \psi_{r_S}([\infty, m]) \Big|_{q=\#k}.$$

This implies

$$\begin{aligned} \int e_{\mu,k}^{-1} &= \prod_{[S]} \left(\sum_{m \geq 0} \psi_{r_S}([\infty, m]) x^{m \dim S} \right) \Big|_{q=\#k} \\ &= \prod_{\substack{\alpha \in \mathbb{N}^I \\ r \geq 1}} \left(\sum_{m \geq 0} \psi_r([\infty, m]) x^{m\alpha} \right) \Big|_{q=\#k}^{s_{\alpha,r}}. \end{aligned}$$

Next

$$\begin{aligned} \prod_{\alpha,r} \left(\sum_{m \geq 0} \psi_r([\infty, m]) x^{m\alpha} \right)^{s_{\alpha,r}} &= \prod_{\alpha,r} \psi_r \left(\sum_{m \geq 0} [\infty, m] x^{m\alpha/r} \right)^{s_{\alpha,r}} \\ &= \prod_{\alpha,r} \psi_r(p(x^{\alpha/r}))^{s_{\alpha,r}} = \prod_{\alpha,r} \psi_r(p(x^\alpha))^{s_{r\alpha,r}} \xrightarrow{\text{Cor. 3.3}} \prod_{\alpha} \text{Pow}(p(x^\alpha), a_\alpha) \\ &= \prod_{\alpha} \text{Exp}(a_\alpha \text{Log}(p(x^\alpha))) \xrightarrow{\text{Lemma 2.3}} \text{Exp} \left(\sum_{\alpha} a_\alpha \frac{x^\alpha}{1-q} \right) = \text{Exp} \left(\frac{a(q)}{1-q} \right). \end{aligned}$$

□

5. IRREDUCIBLE REPRESENTATIONS

In this section, we take the stability map $\theta : I \rightarrow \mathbb{Z}$ to be the zero map. Then the stability of a representation is equivalent to its irreducibility and all representations are semistable. The goal of this section is to give an explicit formula for the generating function $r \in \mathbb{Q}(q)[[x_i | i \in I]]$. In view of Theorem 4.1, this will give us a rather explicit formula for the polynomials $a_\alpha(q)$.

Define the operator T on $\mathbb{Q}(q)[[x_i | i \in I]]$ by $T(x^\alpha) = q^{T(\alpha)} x^\alpha$. In Section 2 we have defined $p = \text{Exp} \left(\frac{\sum_{i \in I} x_i}{1-q} \right)$. We have defined there also the conjugation $\overline{f(q)} = f(q^{-1})$.

Theorem 5.1. *We have $r = \overline{Tp}$. In particular,*

$$\overline{Tp} \circ \text{Exp} \left(\frac{a(q)}{1-q} \right) = 1$$

in the algebra $\mathbb{Q}(q)^{\text{tw}}[[x_i | i \in I]]$.

Proof. Let k be a finite field and $s = \#k$. Recall that

$$r(s) = \sum_{\alpha \in \mathbb{N}^I} \frac{\#\mathcal{R}_\alpha(k)}{\#\mathrm{GL}_\alpha(k)} x^\alpha.$$

We have

$$\#\mathcal{R}_\alpha(k) = s^{\alpha \cdot \alpha - T(\alpha)},$$

where $\alpha \cdot \alpha = \sum_{i \in I} \alpha^i \alpha^i$. Also we have

$$\#\mathrm{GL}_n(\mathbb{F}_s) = \prod_{i=0}^{n-1} (s^n - s^i) = (-1)^n s^{\binom{n}{2}} \prod_{i=1}^n (1 - s^i).$$

This implies

$$\frac{s^{n^2}}{\#\mathrm{GL}_n(\mathbb{F}_s)} = \frac{(-1)^n s^{\binom{n+1}{2}}}{\prod_{i=1}^n (1 - s^i)} = \frac{1}{\prod_{i=1}^n (1 - s^{-i})} = \overline{[\infty, n]} \Big|_{q=s}.$$

We can write now

$$r(s) = \sum_{\alpha \in \mathbb{N}^I} q^{-T(\alpha)} \overline{[\infty, \alpha]} x^\alpha \Big|_{q=s} = \overline{Tp} \Big|_{q=s}.$$

As this equality holds for all finite fields, we get $r = \overline{Tp}$. \square

Remark 5.2. Assume that the quiver does not have oriented cycles. Then the irreducible representations are precisely the one-dimensional ones and we have $a(q) = \sum_{i \in I} x_i$. This implies

$$\mathrm{Exp}\left(\frac{a(q)}{1-q}\right) = \mathrm{Exp}\left(\frac{\sum_{i \in I} x_i}{1-q}\right) = p.$$

It follows that $\overline{Tp} \circ p = 1$. This is a purely combinatorial formula obtained by methods of geometric nature. We give a combinatorial proof of it in Remark 6.4.

6. RECURSIVE FORMULA

Let $a(q) = \sum_{\alpha \in \mathbb{N}^I} a_\alpha(q) x^\alpha$ be the generating function for the irreducible representations of a quiver. In this section we will show that the expression

$$\frac{a(q) - \sum_{i \in I} x_i}{1-q}$$

is well defined at $q = 1$ and we will give a recursive formula for the computation of both the expression and of its value at $q = 1$.

Define the Ringel matrix R by the formula $\langle \alpha, \beta \rangle = \alpha^t R \beta$, for any $\alpha, \beta \in \mathbb{N}^I$. Then

$$R_{ij} = \delta_{ij} - \#\{\text{arrows from } i \text{ to } j\}.$$

For any $f \in \mathbb{Q}(q)[[x_i, i \in I]]$, we denote by f_α the coefficient of x^α in f .

Theorem 6.1. *Expression $f = \mathrm{Exp}\left(\frac{a(q) - \sum x_i}{1-q}\right)$ can be determined by the recursive formula:*

$$(p(-R\alpha) \cdot f)_\alpha = 0$$

for any $\alpha > 0$. This expression is well defined at $q = 1$. The specialization $f|_{q=1}$ can be determined from the same recursive formula, using the simplification $p(\lambda)|_{q=1} = \prod_{i \in I} (1 - x_i)^{-\lambda^i - 1}$.

Proof. We have shown that

$$\overline{Tp} \circ \text{Exp} \left(\frac{a(q)}{1-q} \right) = 1.$$

Let us denote $\text{Exp} \left(\frac{a(q)}{1-q} \right)$ by g . Then, for any $\alpha > 0$, we have

$$\sum_{0 \leq \beta \leq \alpha} (\overline{Tp})_\beta g_{\alpha-\beta} q^{-\langle \beta, \alpha - \beta \rangle} = 0.$$

Note that

$$(\overline{Tp})_\beta = q^{-T(\beta)} \overline{p}_\beta = q^{-\langle \beta, \beta \rangle} \overline{p}_\beta.$$

This implies

$$\begin{aligned} 0 &= \sum_{0 \leq \beta \leq \alpha} \overline{p}_\beta g_{\alpha-\beta} q^{-\langle \beta, \alpha \rangle} = \sum_{0 \leq \beta \leq \alpha} g_{\alpha-\beta} q^{-\beta^t R \alpha} \overline{p}_\beta \\ &= \sum_{0 \leq \beta \leq \alpha} g_{\alpha-\beta} (S_{-R\alpha} \overline{p})_\beta = (g S_{-R\alpha} \overline{p})_\alpha. \end{aligned}$$

By Lemma 2.4, we have $S_{-R\alpha} \overline{p} = p(-R\alpha)/p$ and therefore

$$(p(-R\alpha) \cdot g/p)_\alpha = 0$$

for any $\alpha > 0$. We note that

$$g/p = \text{Exp} \left(\frac{a(q) - \sum x_i}{1-q} \right) = f.$$

The above formula allows us to determine f recursively. Moreover, as the expressions $[n, k]$ are well defined at $q = 1$, we deduce that the $p(\lambda)$ are well defined at $q = 1$. The recursive formula implies that $f = g/p$ is well defined at $q = 1$. \square

Corollary 6.2. *Let (Γ, I) be a quiver with one node and m loops. Let $a = \sum_{d \geq 0} a_d x^d$ be the generating function of the polynomials counting absolutely irreducible representation of Γ over finite fields. Then*

$$\text{Exp} \left(\frac{a(q) - x}{1-q} \right) \Big|_{q=1} = 1 - mx.$$

Proof. To derive the statement from our recursive formula, we have to prove that for any $n > 0$, the coefficient of x^n in

$$(1-x)^{n-mn-1} (1-mx)$$

is zero. This can be checked directly. \square

Remark 6.3. The above corollary is equivalent to the formula

$$\frac{a_d(q)}{q-1} \Big|_{q=1} = \frac{1}{d} \sum_{k|d} \mu(d/k) m^k, \quad d \geq 2.$$

proved in [10]. Indeed, as $a_1(q) = q^m$, we deduce from the formula

$$\begin{aligned} \frac{a(q) - q^m x}{q-1} \Big|_{q=1} + mx &= \sum_{d \geq 1} \frac{1}{d} \sum_{k|d} \mu(d/k) m^k x^d \\ &= \sum_{k, l \geq 1} \frac{\mu(l)}{l} \psi_l \left(\frac{m^k x^k}{k} \right) = \sum_{l \geq 1} \frac{\mu(l)}{l} \psi_l \left(\log \left(\frac{1}{1-mx} \right) \right) = -\text{Log}(1-mx). \end{aligned}$$

Note that $mx = \frac{q^m x - x}{q - 1} \Big|_{q=1}$. This implies

$$\frac{a(q) - x}{1 - q} \Big|_{q=1} = \text{Log}(1 - mx).$$

The last formula is equivalent to the statement of the corollary.

Remark 6.4. We can give now a combinatorial proof of the formula $\overline{Tp} \circ p = 1$ in the case when the quiver does not have oriented cycles. By the proof of the above theorem, this is equivalent to the condition

$$(p(-R\alpha) \cdot p/p)_\alpha = 0$$

for any $\alpha > 0$. Thus, we have to show that

$$p(-R\alpha)_\alpha = [-R\alpha, \alpha] = 0$$

for $\alpha > 0$. We can assume that R is upper-triangular with ones on the diagonal after permuting the nodes of the quiver. The above expression is a product of q -binomial coefficients and it is easy to see that one of them has the form $[-n, n]$ for some $n > 0$. But $[-n, n] = 0$ for $n > 0$.

Remark 6.5. Expanding the polynomial $a_d(q)$ in powers of $q - 1$, Remark 6.3 shows in particular that the linear term

$$\frac{1}{d} \sum_{k|d} \mu(d/k) m^k$$

in this expansion is nonnegative (the constant term being zero except for the trivial case $d = 1$): this number is well-known in combinatorics as the number of primitive necklaces consisting of d beads in m colours (see e.g. [13]).

It has been verified by computer experiments, in the cases $m = 2, 3, 4$ and $d \leq 12$, that indeed *all* coefficients in this expansion are nonnegative. One might therefore conjecture that

$$a_d(q) \in \mathbb{N}[q - 1]$$

for all $d \geq 1$.

Even more optimistically, one might hope for a geometric explanation for such a result, for example the existence of a finite stratification of the moduli space of d -dimensional irreducible representations (see [10] for the precise definition) into strata isomorphic to tori (and thus contributing powers of $q - 1$ to the number of rational points of the moduli space).

The string representations of the m -loop quiver, introduced in [10] can be seen as a first step in this program: they are parametrized up to isomorphism by primitive necklaces as above, and thus provide an explanation for the linear term $\frac{1}{d} \sum_{k|d} \mu(d/k) m^k$ in the above expansion.

Against this background, it is natural to study the expansion

$$\text{Exp} \left(\frac{a(q) - x}{1 - q} \right) \Big|_{q=1} = \sum_{n=0}^{\infty} f_n(m, t) (q - 1)^n.$$

Besides our result saying that $f_0(m, t) = 1 - mt$, computer experiments suggest

$$f_1(m, t) = \binom{m}{2} \frac{t(t-1)}{(1-mt)^2}$$

and more generally

$$f_n(m, t) \cdot (1 - mt)^{3n-1}$$

is a polynomial in m and t having degree $3n - 1$ with respect to t .

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